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SOME ASPECTS OF GALACTIC DYNAMICS IN THE GAIA ERA

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Abstract. In the context of this school on stellar physics with Gaia, I briefly present some basic tools of theoretical Galactic dynamics, both in terms of the construction of equilibrium models and of linear perturbation theory.

Keywords: Milky Way dynamics; Gaia

1 Introduction

The Gaia mission (Gaia Collaboration et al. 2016) has been (and still is) an extraordinary milestone in offering full six-dimensional phase-space information for a large number of stars of the Milky Way over a larger volume than ever before (Gaia Collaboration et al. 2023). This offers a unique possibility to improve our dynamical models of the Galaxy, which can serve as a laboratory for a plethora of unanswered questions in the field of galaxy formation and evolution, for instance in terms of understanding the role of environment, the accretion history, the nature of internal instabilities such as bars and spiral arms, or even test models for the nature (or existence) of dark matter.

The most common typically "top-down" dynamical approach consists in generating *ab initio* simulations of galaxies resembling the Milky Way in a cosmological context (e.g., Renaud et al. 2021). While extremely valuable for understanding general features of galaxy formation, this method nevertheless lacks the flexibility needed to create a model that precisely aligns with the vast and detailed data available for our own Galaxy. The complementary "bottom-up" approach for dynamical modeling consists in starting from existing Galactic data, and constructing a model from there rather than relying on simulations. In such a model, a single-particle phase-space distribution function is used to represent all the different constituent particles, namely various stellar populations and dark matter. Such model-building typically starts with the assumptions of dynamical equilibrium and axisymmetry. These assumptions allow us to make use of Jeans' theorem constraining the distribution function to depend only on three integrals of motion, which can typically be chosen to be the radial, azimuthal, and vertical action variables of the canonical action-angle phase-space coordinates. Such dynamical models are however largely insufficient to describe our Galaxy, which is evident from its non-axisymmetric nature featuring, e.g., a prominent central bar and spiral arms that have long been known to leave their imprint in the phase-space structure of the Milky Way (e.g., Famaey et al. 2005). Additionally, recent insights from Gaia have revealed a vertical disequilibrium of the Galactic disk (Antoja et al. 2018), possibly linked to a complex interplay between external disturbances and internal non-axisymmetries (Li et al. 2023). This however does not invalidate the usefulness of the approach, since all these effects can in principle be treated, to a certain extent, through perturbation theory. In this short lecture, I lay out the general principles underlying this approach and present a few recent results along the way.

2 Fundamental equations of Galactic Dynamics

The evolution of any N-particle system (with N being for instance the number of stars, or even of stars and dark matter particles, in a galaxy) is characterized by the conservation of the probability distribution function (DF) in a phase space of $6N$ dimensions, F_N, according to Liouville's theorem. This phase-space DF in $6N$ dimensions is strictly conserved along trajectories following the Hamiltonian flow of the system. The Bogoliubov (or BBGKY) hierarchy then relates the n-particle (with $n < N$) DF to the $n+1$ DF. The single-particle phase space density $F_1(\mathbf{x}, \mathbf{v}, t)$ (or just $F(\mathbf{x}, \mathbf{v}, t)$ hereafter) is then related to the two-particle DF only through a correlation integral term, which goes to zero in the large N and large relaxation time limit typical for gravity in galaxies (Binney & Tremaine 2008). The single-particle DF then obeys the Vlasov (or collisionless Boltzmann) equation, expressing conservation of particles in infinitesimal patches along the phase space trajectory $\{\mathbf{x}(t), \mathbf{v}(t)\}$. Coupled with the equation for the gravitational potential Φ in the weak-field limit, namely Poisson's equation, this leads to the fundamental system of equations for Galactic dynamics (or at least, and more broadly, for collisionless stellar dynamics), the Vlasov-Poisson system of equations:

$$\begin{cases} \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \mathbf{v} \cdot \frac{\partial F}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial F}{\partial \mathbf{v}} = 0, \\ \nabla^2 \Phi = 4\pi G \rho = 4\pi G \int \mathrm{d}^3\mathbf{v}\, F. \end{cases} \tag{1}$$

The single-particle Hamiltonian is simply written as $H = \frac{v^2}{2} + \Phi(\mathbf{x})$. Things become a bit more complicated when including a gas component, but this is beyond the scope of the present lecture, and gas will typically be treated as an external component to the Galactic gravitational potential Φ.

3 Equilibrium models

In a realistic axisymmetric and time-independent Galactic potential, the majority of orbits are regular or quasi-regular, exhibiting quasiperiodic patterns in the sense that their Fourier transforms have only discrete frequencies that are integer linear combinations of three fundamental frequencies. These orbits therefore possess three isolating integrals of motion, which uniquely define them within the Galaxy's gravitational potential. The Liouville-Arnold theorem then states that *if, in a Hamiltonian dynamical system with N degrees of freedom, there are also N independent, Poisson commuting integrals of motion, then there exists a canonical transformation to coordinates in which the transformed Hamiltonian is dependent only upon the new generalized momenta, and their canonically conjugated variables evolve linearly in time.* These are called the action-angle variables $(\mathbf{J}, \mathbf{\Theta})$, where the actions $\mathbf{J}$ (new generalized momenta) are integrals of the motion and the angles $\mathbf{\Theta}$ evolve linearly with time. Said otherwise, Hamilton's equations simply become:

$$\begin{cases} \frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{\Theta}} = 0, \\ \frac{\mathrm{d}\mathbf{\Theta}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{J}} = \mathbf{\Omega}(\mathbf{J}). \end{cases} \tag{2}$$

According to Jeans' theorem, the equilibrium stellar phase-space DF, $F(\mathbf{x}, \mathbf{v})$, for any Galaxy component should then depend solely on the actions, $F = F(\mathbf{J})$. While analytical relations between action-angles $(\mathbf{J}, \mathbf{\Theta})$ and usual phase-space coordinates $(\mathbf{x}, \mathbf{v})$ are seldom attainable for most potentials Φ, these variables present numerous advantages: in an equilibrium configuration, the stars' angles, $\mathbf{\Theta}$, are uniformly distributed on phase-space orbital tori defined solely by $\mathbf{J}$, and the phase-space density of stars, $F(\mathbf{J})\mathrm{d}^3\mathbf{J}$, directly relates to the number of stars dN in an infinitesimal action range, divided by $(2\pi)^3$; additionally, the actions remain adiabatically invariant during a gradual change in the Galactic potential; finally, as we shall see in the next section, these variables serve as natural coordinates for perturbation theory. Notwithstanding the absence of an easy analytical transformation for most potentials, in order to transform from actions and angles to positions and velocities, one typically first expresses the Hamiltonian in the action-angle coordinates $(\mathbf{J}_T, \mathbf{\Theta}_T)$ of a *toy* potential, for which the transformation to positions and velocities is fully known analytically (generally with an isochrone potential). One then searches for a type 2 generating function $G(\mathbf{\Theta}_T, \mathbf{J})$ expressed as a Fourier series expansion on the toy angles $\mathbf{\Theta}_T$, for which the Fourier coefficients are such that the Hamiltonian remains constant at constant $\mathbf{J}$. This generating function fully defines the canonical transformation from actions and angles to positions and velocities. This method is known as the 'Torus mapping' (McGill & Binney 1990). For the inverse transformation, an estimate based on separable potentials can be used. These potentials are known as Stäckel potentials (e.g. Famaey & Dejonghe 2003), for which each generalized momentum depends on its conjugated position through three isolating integrals of the motion. These potentials are expressed in spheroidal coordinates defined by a focal distance. For a Stäckel potential, this focal distance of the coordinate system is related to the first and second derivatives of the potential. One can thus use the true potential at any configuration space point to compute the equivalent focal distance *as if* the potential were of Stäckel form, and compute the corresponding isolating integrals of the motion and the actions. This method is known as the Stäckel fudge.

Both transformations are part of the Action-based Galaxy Modelling Architecture (AGAMA; Vasiliev 2019) code. Other methods relying on unsupervised machine learning methods have also been developed (Ibata et al. 2021).

To construct an equilibrium model, one can then devise parametric DFs representing the different components of the Galaxy. Typically, one can (i) start from a guess for the Galactic potential (already taking into account some constraints such as the rotation curve, and including a gas surface density) and for the action-space DF expressed as a linear combination of different components (young thin disk, intermediate-age thin disk, old thin disk, thick disk, stellar halo, dark matter halo, bulge/fat disk in the central regions,...), (ii) compute the configuration space density associated with this guess-DF when integrating it over velocity space, (iii) compute a new Galaxy potential from this density with Poisson equation, (iv) take a weighted mean of this newly computed potential and the original guess-potential, and iterate until convergence. This procedure ensures one to construct a self-consistent equilibrium model. In principle, the likelihood of such a model can then be obtained on an individual star-by-star basis in observables space (sky positions, parallaxes, proper motions from Gaia, radial velocities from the RVS or ground-based surveys if available) taking into account the selection function, and the whole parameter space can be explored fully. Interestingly, taking away the baryonic component also allows to compute the shape of the dark matter halo without the contraction related to the presence of baryons, which can give clues to the nature of dark matter. Such a titanesque procedure has however not been attempted yet on the full Gaia dataset, but the closest to this endeavour has been achieved by Binney & Vasiliev (2023), who presented a qualitative fit, and Binney & Vasiliev (2024) who fit a small subset of stars with APOGEE spectroscopy, including an additional dependence on [Fe/H] and [Mg/Fe] in the DF. These models reveal in detail the zeroth order orbital structure of our Galaxy in action space.

4 Perturbation theory

Such equilibrium axisymmetric models are however insufficient to describe the current phase-space structure of the Galaxy, which has clear imprint of non-axisymmetries. Understanding these is particularly important also in view of understanding the secular evolution of the Galaxy, whose main internal driving mechanisms are instabilities leading to non-axisymmetric modes (bar, spiral arms). To take these into account, let us now consider the influence of a small perturbation to the Hamiltonian,

$$\Delta H(\mathbf{x},t) = \psi(\mathbf{x},t) \ll \Phi. \tag{3}$$

In action-angle coordinates, such a perturbing potential is 2π-periodic in the angles and can therefore typically be written as a Fourier series over the angles

$$\Delta H(\mathbf{J},\boldsymbol{\Theta},t) = \sum_{\mathbf{n}} \Delta H_{\mathbf{n}}(\mathbf{J},t) \exp(\mathrm{i}\mathbf{n}.\boldsymbol{\Theta}) \tag{4}$$

The total perturbed DF can then be expressed as $F(\mathbf{J}) + f(\mathbf{J},\boldsymbol{\Theta})$ with the response DF, $f \ll F$, obeying the *linearized* collisionless Boltzmann equation, obtained by dropping all the high-order terms in the original Vlasov equation:

$$\frac{\partial f}{\partial t} + \boldsymbol{\Omega}\cdot\frac{\partial f}{\partial \boldsymbol{\Theta}} - \frac{\partial F}{\partial \mathbf{J}}\cdot\frac{\partial \Delta H}{\partial \boldsymbol{\Theta}} = 0. \tag{5}$$

Expressing the response DF as a Fourier series over the angles,

$$f(\mathbf{J},\boldsymbol{\Theta},t) = \sum_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{J},t) \exp(\mathrm{i}\mathbf{n}.\boldsymbol{\Theta}), \tag{6}$$

and assuming that the response is zero at $t=0$, each Fourier coefficient is then:

$$f_{\mathbf{n}}(\mathbf{J},t) = \mathrm{i}\,\mathbf{n}\cdot\frac{\partial F}{\partial \mathbf{J}}\int_0^t \mathrm{d}\tau\, \Delta H_{\mathbf{n}}(\mathbf{J},\tau) \exp[-\mathrm{i}\,\mathbf{n}\cdot\boldsymbol{\Omega}\,(t-\tau)]. \tag{7}$$

Let us consider that the Fourier coefficients of the perturbing potential depend on time only through a logistic function controlling the amplitude of the perturbation and through a periodic sinusoidal function of frequency ω_p, which can account for a perturbing potential of m-fold symmetry rotating with a fixed pattern speed $\Omega_p = -\omega_p/m$. Then, at $t=\infty$, we have (e.g., Monari et al. 2016; Al Kazwini et al. 2022):

$$f_{\mathbf{n}} = \Delta H_{\mathbf{n}} \frac{\mathbf{n}.\frac{\partial F}{\partial \mathbf{J}}}{\mathbf{n}.\boldsymbol{\Omega} + \omega_p}. \tag{8}$$

Knowing the perturbing potential, one can thus compute the response DF in this way. For instance, in Monari et al. (2016), using the epicyclic approximation to get an analytic relation between the action-angle variables and the positions and velocities, and considering a 3D spiral arm perturber with corotation in the outer Galaxy, we showed that the first order moments of the perturbed DF describe "breathing" modes of the Galactic disc in perfect accordance with simulations. But obviously the denominator in Equation 8 diverges at resonances (problem of small divisors). This is due to the fact that, close to resonances, the orbital tori are radically modified and the linear perturbation theory breaks down. It is however possible to circumvent this problem when only one main perturber is at play: for each resonance, one can define a new set of actions and angles to describe the orbits, through two consecutive canonical transformations in order to find the relevant action variables to use in the resonant region (Monari et al. 2017). One can then populate the new tori by phase-averaging the original unperturbed DF $F(\mathbf{J})$ over these new resonant tori, which reproduces reasonably well what happens in simulations. However, this method breaks down in the presence of multiple perturbing patterns because of chaos related to resonance overlaps (Minchev & Famaey 2010), which have otherwise interesting consequences in terms of secular evolution of the disk through their effect on heating and radial migration. Radial migration can in turn be probed through the chemo-orbital distribution of equilibrium models (see previous section), which in principle can help constrain the non-axisymmetric patterns driving it: for instance, the presence of super-metal rich stars in the Solar neighborhood (Kordopatis et al. 2015) is a powerful chemo-orbital probe of the effect of past spiral perturbations and their coupling with the bar. To circumvent the problem of resonance overlaps in linear perturbation theory, one can resort to the backwards integration method originally developed by Vauterin & Dejonghe (1997), based on the conservation of the DF in infinitesimal phase-space patches following the Hamiltonian flow, encoded in the Vlasov equation. One can then compute the current DF $F(\mathbf{x}, \mathbf{v})$ by integrating orbits backward in time to an axisymmetric equilibrium state $F(\mathbf{J})$. Using this method, most kinematic groups observed in the vicinity of the Sun can be reproduced with a multi-modal bar model (Monari et al. 2019), but in regions beyond the Sun, the average radial velocity obtained from such a bar-only model clearly indicates the necessary inclusion of spiral arms (Khalil et al. 2023). The challenge is, however, to recover the velocity field measured with Gaia DR3 without destroying the already pretty satisfactory local velocity field in the presence of the bar alone. This is work in progress, and should shed light on the present-day dynamical structure (pattern speed etc.) of spiral arms in the Milky Way.

5 Self-consistency

In the previous section, I considered the response to a perturbation treated as an external one, even when dealing with an internal one such as a spiral pattern or the bar. While this can catch some highly interesting features of the response, especially relatively far away from the main density perturbation itself, it is not a self-consistent procedure. For instance, it does not allow to follow self-consistently the growth and/or decay of an instability, or to treat the self-consistent response of the Milky Way to an external perturber, for instance to the Sgr dwarf or to the Large Magellanic Cloud (LMC), which are both influencing the current dynamical state of the Galaxy.

For such an endeavour, the perturbation to the Hamiltonian considered in the previous section can be divided into an external ψ^{e} and a self-consistent ψ^{s} internal part:

$$\Delta H(\mathbf{x}, t) = \psi^{\mathrm{e}}(\mathbf{x}, t) + \psi^{\mathrm{s}}(\mathbf{x}, t). \tag{9}$$

When considering an external perturber that can itself accelerate the reference frame of the Galaxy, which is the case when considering an infalling LMC with a mass of $1.8 \times 10^{11} \mathrm{M}_{\odot}$, one can also add to the perturbing Hamiltonian a potential term accounting for a pseudo-force associated to this accelerated frame (Rozier et al. 2022). One can also consider as an "external" perturber some simple shot-noise perturbation to the disk, which disappears quickly but lets the bar and spiral instabilities develop self-consistently as a consequence.

With a perturbation given by Equation 9, the solution to the linearized collisionless Boltzman equation is still given at any time by Equations 6-7 hereabove, but now at each time one computes the response potential via the integral of f over velocity space through Poisson's equation, so that we can update the perturbing Hamiltonian self-consistently. But now we have a problem: the linearized collisionless Boltzmann equation is best expressed in action-angle phase-space coordinates, but we also need to solve the Poisson equation to get the potential of the response, and the latter is best solved in configuration space coordinates, not actions and angles. One can bypass this difficulty by resorting to Kalnajs trick (Kalnajs 1977): projecting everything on a bi-orthogonal basis of potentials $\psi^{(\mathbf{p})}$ and densities $\rho^{(\mathbf{p})}$ that solve the Poisson equation once and for all. The choice of the bi-orthogonal basis obviously depends of the problem at hand. For instance, when computing the self-consistent response of the dark and stellar halos to the LMC infall onto the Milky Way, one can simply choose spherical harmonics multiplied by a radial component of Clutton-Brock form (Clutton-Brock 1972; Rozier et al. 2022). Another basis must be used when treating the response of the Galactic disk.

The key aspect of a bi-orthogonal potential-density basis is that it obeys the bi-orthogonality condition:

$$\int \mathrm{d}\mathbf{x}\, \psi^{(\mathbf{p})}(\mathbf{x})\, \rho^{(\mathbf{q})*}(\mathbf{x}) = -\delta_{\mathbf{p}}^{\mathbf{q}}, \tag{10}$$

where $*$ indicates the complex conjugate while $(\mathbf{p})$ and $(\mathbf{q})$ stand for multiplets of indices, for instance in the Clutton-Brock case triplets of indices (l, m, n), where l and m are just spherical harmonic degree and order, and n is the radial order. Note that this inner product could be interpreted as proportional to the interaction potential energy between two disturbances, which is thus zero between two distinct basis elements.

Now, in configuration space, the perturber and the self-consistent response can just be expressed as vectors in this basis of potentials and densities, which we call vectors $\mathbf{b}(t)$ and $\mathbf{a}(t)$ respectively. Thanks to the bi-orthogonality condition, in order to obtain the vector decomposition of the self-gravitating response, one just needs to take the inner product of the perturbed density and each of the basis elements:

$$a_{\mathbf{p}}(t) = -\int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t)\, \psi^{(\mathbf{p})*}(\mathbf{x}), \tag{11}$$

This can be canonically transformed into an integral over actions and angles, in which one simply needs to insert the solution of the linearized collisionless Boltzmann equation for f, given by Equations 6-7. The solution for $a_{\mathbf{p}}(t)$ then simply becomes an integral over time, in which everything that depends on the background state $F(\mathbf{J})$ is absorbed into a "*response matrix*" that multiplies the vector $\mathbf{b} + \mathbf{a}$ (see Rozier et al. 2022, for details). As an example, one can for instance reconstruct the LMC with 3216 basis elements, and study the response of an isotropic halo to its infall, in 20 timesteps over 2 Gyr (Rozier et al. 2022). Contrary to the case of N-body simulations, here the system is still following orbits (and is still responding to the perturber) between timesteps. An interesting aspect of the response matrix specific to non-rotating spheres is that there is no coupling between different angular harmonics: each harmonic in the response is only induced by the corresponding harmonic in the perturber, mediated by that same harmonic in the matrix. This allows one to make a harmonic decomposition of the response for different halo anisotropies. In Rozier et al. (2022), we found that the large scale over/underdensity induced by the LMC in the stellar halo corresponds to the $l = 1$ terms, and is purely associated to the reflex motion: the underlying kinematics of the halo do not change this. What is sensitive to the underlying kinematics of the stellar halo population one considers is the local wake around the LMC. The response is much stronger in the case of a radially-biased halo. The orientation and winding of the quadrupolar ($m = 2$) response is also very different. Importantly, the response of the stellar halo does not allow to infer the presence of a particle-made dark matter halo. We also looked at the effect of the different Fourier numbers in our Fourier decomposition to see which resonance dominates in different anisotropic cases. For a radially anisotropic stellar halo, the inner Lindblad resonance dominates: in that case, instead of attracting particles which can move with it, the LMC rather attracts orbits which can precess with it, the wake is relatively slow, trails behind the LMC, and remains strong. For a tangentially anisotropic stellar halo, we found that the corotation frequency dominates: the wake is then basically following the LMC and dissipates quickly as the LMC moves inducing a weaker response. These insights on the dynamics at play can only be gotten from such an analytical treatment, which is thus very complementary to simulations. Similar studies should, in the future, be carried out on the response of the Galactic disk to its interaction with the Sgr dwarf, whose signature might be partially encoded in the phase-spiral discovered in Gaia data.

6 Conclusions

Our Galaxy can serve as a laboratory to answer a plethora of yet unanswered questions in the field of galaxy formation and evolution. In this brief presentation, I outlined the basics of theoretical/analytical Galactic dynamics, both in terms of the construction of equilibrium models and of linear perturbation theory, which are very powerful tools to dissect the dynamical mechanisms at play (which are often difficult to disentangle in numerical simulations), and are in this sense highly complementary to numerical simulations of galaxy formation in a cosmological context. I also gave a few examples of recent results using such tools. Constructing equilibrium models reveals the zeroth order axisymmetric orbital structure of the Galaxy, and can be coupled with a chemical decomposition to reveal the formation history of its different stellar components. This chemical decomposition of phase-space already reveals that the disk of the Galaxy is constantly reshuffling its orbits under internal secular evolution processes, such as stellar radial migration, driven by non-axisymmetric (short-lived and/or long-lived) modes and their respective couplings. Understanding the exact nature of spiral arms and of their role in the secular evolution of the Galaxy has remained a surprisingly pressing challenge for several decades, and a chemo-orbital analysis of stars of the Milky Way is a powerful probe to advance on this question. Gaia data, complemented by ground-based spectroscopic surveys, have also revealed a present-day highly complex picture of our Galaxy, both affected by internal instabilities (spirals, bar) and external perturbations (LMC, Sgr dwarf). Here again, dynamical modeling and linear perturbation theory are powerful complementary tools to numerical simulations in order to disentangle the various effects at play, and reconstruct past events such as the Gaia-Sausage-Enceladus merger. Some of these effects at play (possible slow-down of the rotation of the Galactic bar, dynamical friction on the Sgr dwarf) also involve – in principle – interactions of the stellar components of the Galaxy with the dark matter halo, providing a potentially powerful probe of the nature of dark matter. More generally, dynamical modeling can probe the shape and substructure of the putative dark matter halo, and compare it with expectations from the standard ΛCDM model and in alternative frameworks. Regarding the formation history of the Galaxy, many questions still remain unanswered too, related to, e.g., the origin of the chemical thick disk, or the nature of the disk-halo interface. For these questions, the tools presented here complemented by future better and higher resolution measurements of stellar chemical abundances and of more precise stellar ages (e.g., Miglio et al. 2017) should allow us to be confident that the complex formation history of our Milky Way will be revealed in full in the coming decades.

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